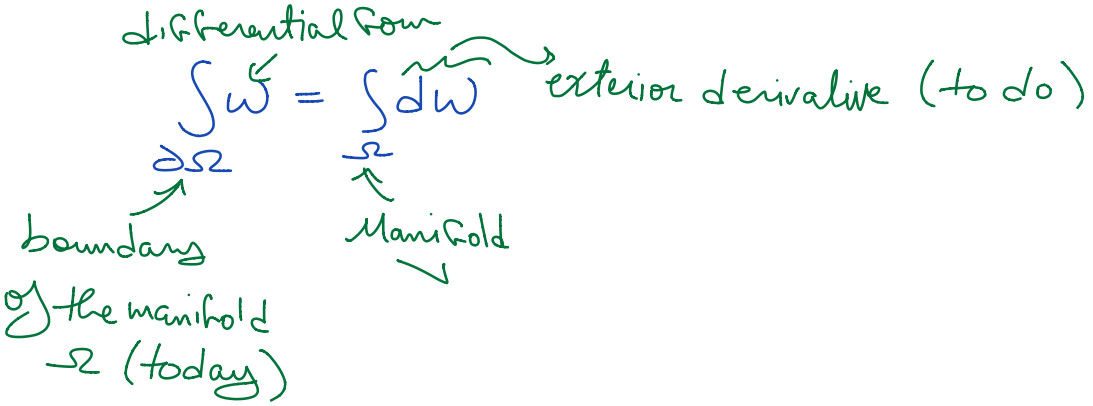
Boundary orientation

One of our main goals, to generalize the FTOC, will give us stoke's theorem, which informally states.

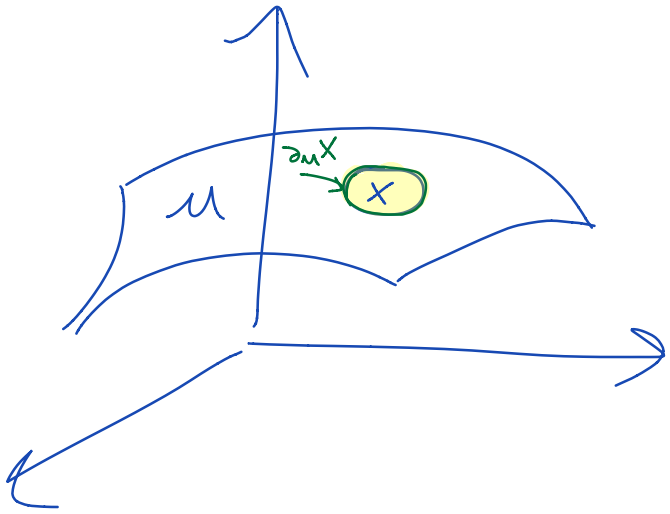


Our goal for today will be to understand what the boundary of a manifold is & how to orient it.

Definition (boundary)

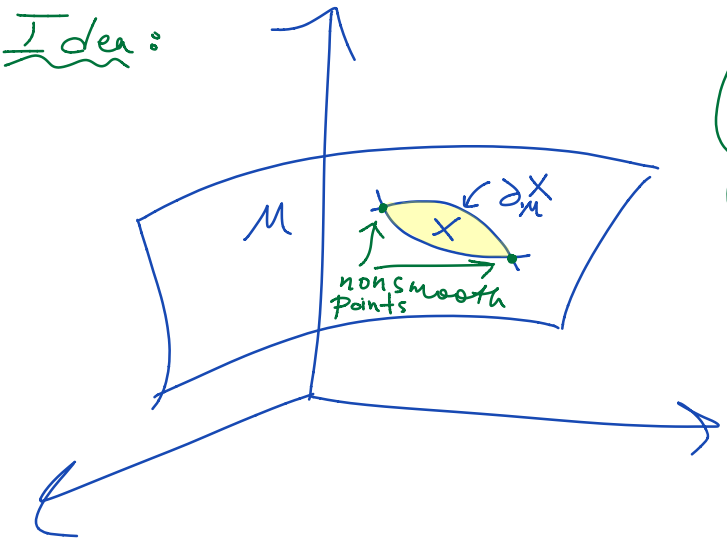
Let $M \subset \mathbb{R}^n$ be a k-manifold and let $X \subset M$.

The boundary of X in M is denoted $\partial_M X$. It is the set of points $x \in X$ such that every neighborhood of x contain points in X & points in $M - X$.



We will have to be careful about "non-smooth" points of the boundary

Idea:



See book for precise definition

The set of smooth pts of $\partial_M X$ is called the smooth boundary of X and denoted $\partial_M^S X$

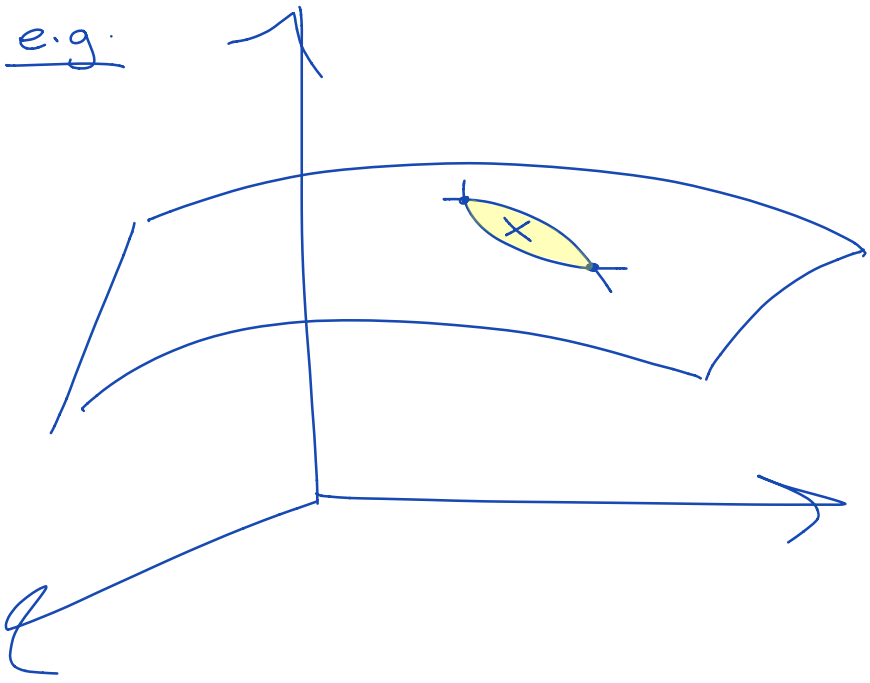
Proposition: $\partial_M^S X$ is a smooth $k-1$ manifold

Definition: A Piece-with-boundary of the k-manifold M is a compact subset $X \subset M$ such that

(1) The set of non-smooth points in $\partial_M X$ has $(k-1)$-dimensional volume $= 0$

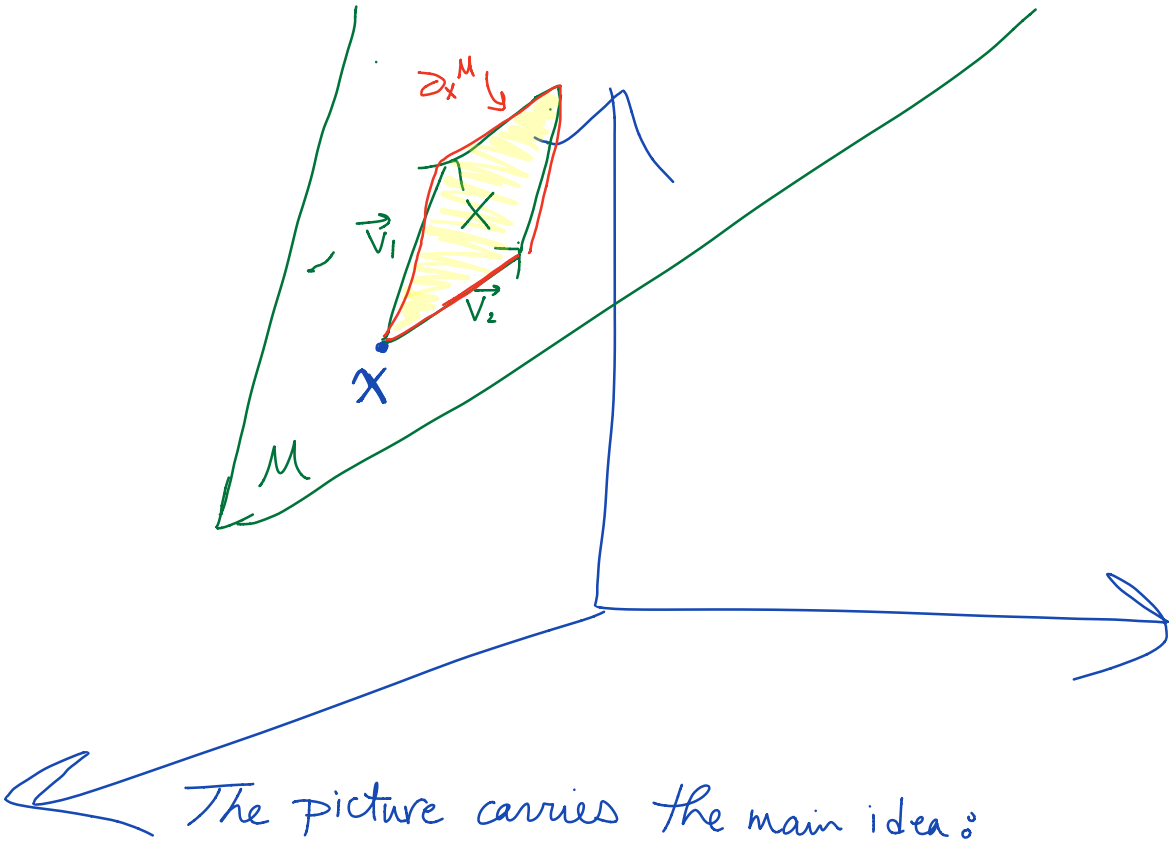
(2) $\mathrm{Vol}_{k-1}(\partial_M^s X) < \infty$

e.g.



Example: A k-parallelogram in $\mathbb{R}^n$ is a piece with boundary.

anchored at $\vec{x}$



The picture carries the main idea:

- M is the subspace spanned by $\vec{v}_1 - \vec{v}_k$, but shifted by x.
- X is the k-parm
- the non-smooth boundary (pts in $\mathbb{R}^2$, lines in $\mathbb{R}^3$, ...) is a $k-2$ manifold $\Rightarrow$ its $\text{Vol}_{k-1} = 0$
- the smooth boundary is a union of $k-1$ parm's $\Rightarrow$ has finite Vol_{k-1}.

Theorem: If $X \subset M$ is a k-dimensional piece with boundary then $\mathrm{Vol}_k X < \infty$

Fact: If $X \subset M$ (M: smooth k-manifold.) is a PWB then $\forall$ invertible $n \times n$ matrix and every $\vec{c} \in \mathbb{R}^n$

$$AX + \vec{c} \text{ is a PWB of } \underbrace{AM + \vec{c}}_{\text{still a } k\text{-manifold}}$$

— × —

Orientation of Piece with boundary

The orientation of $X \subset M$ will just be the orientation of M
(or of a big enough piece of M containing X)

Example



Orienting $P_x(\vec{v}_1, \dots, \vec{v}_k)$

- The standard orientation of $P_x(\vec{v}_1 \dots \vec{v}_k)$ is the orientation where

$\vec{v}_1 \dots \vec{v}_k$ form a direct basis

of $T_y M \quad \forall y \in M$.

Remark: From now on

When we refer to $P_x(\vec{v}_1, \dots \vec{v}_k)$

we mean, with the standard orientation.

(otherwise $-P_x(\vec{v}_1 \dots \vec{v}_k)$)

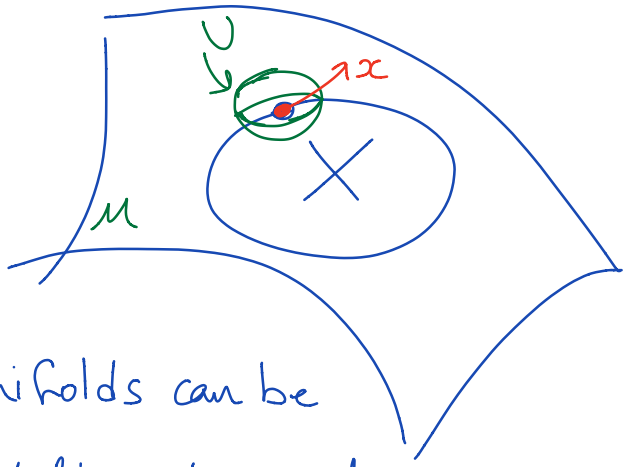
Boundary orientation

Since our goal is to integrate

$$\int_{\partial\Omega} w$$

and we have now figured out that $\partial\Omega$ is a $k-1$ manifold, we still need to orient it.

Recall:



(around $x \in M$), manifolds can be described as the set of solutions to equations

$$f : U \to \mathbb{R}^{n-k}$$

(C^1; U: ball around x)

$$M \cap U = \{ z : f(z) = 0 \}$$

where $[Df(x)]$ is surjective.

- Since the smooth boundary is a ($k-1$ manifold) manifold

if $x \in \partial_M X$, $\exists\, g : U \to \mathbb{R}$,

with $\left[D\begin{pmatrix} \vec{f} \\ g \end{pmatrix}\Big|_x \right]$ surjective

($U \to \mathbb{R}^{n-k+1}$)

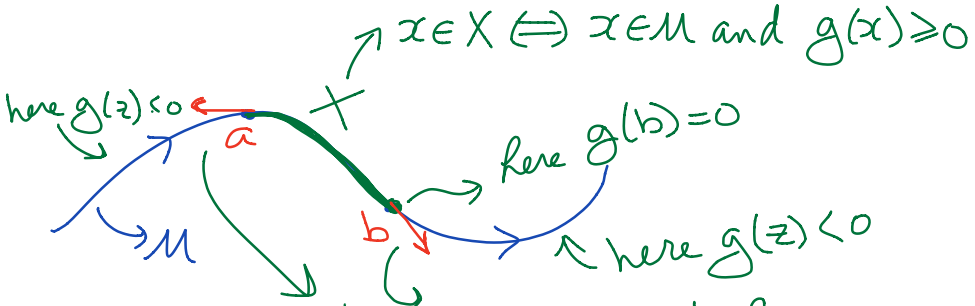
and $\begin{pmatrix} \vec{f} \\ g \end{pmatrix}(x) = 0$ & $X \cap U = \{ z : f(z) = 0,\ g(z) \geq 0 \}$

(ball around X; f: describes M; g: describes X)

Defn If X is a PWB of M and $x \in \partial_M^s X$ and $\vec{v} \in T_x M$ then $\vec{v}$

- points outward from X if $[Dg(x)]\vec{v} < 0$
- points inwards if $[Dg(x)]\vec{v} > 0$

Example



the red arrow points from "$g = 0$" to "$g < 0$"

so $[Dg(x)]\vec{v}$ is negative (this is the directional derivative)

Defⁿ: Let M be a k-manifold oriented by Ω and let P be a PWR of M. Let $x \in \partial_M^s P$ and $\vec{v}_{out} \in T_x M$ be outward pointing

Then $\Omega^{\partial}_{\vec{x}}$:

$$\Omega^{\partial}_{\vec{x}}(\underbrace{\vec{v}_1, \dots, \vec{v}_{k-1}}_{\text{basis for } T_x \partial_M^s P}) = \Omega_{\vec{x}}(\underbrace{\vec{v}_{out}, \vec{v}_1, \dots, \vec{v}_{k-1}}_{\text{basis for } T_x M})$$

defines an orientation for $\partial_M^s P$.

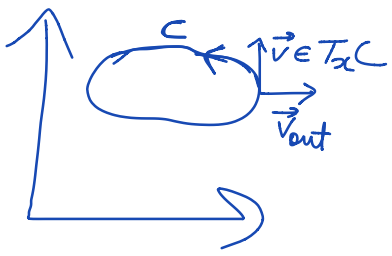
Example: In our previous example

If $\vec{v}_{out}$ points "in the direction of the orientation", $\Omega^{\partial}_x = +1$

otherwise $\Omega^{\partial}_x = -1$

So: $\Omega^{\partial}_b = +1$ & $\Omega^{\partial}_a = -1$

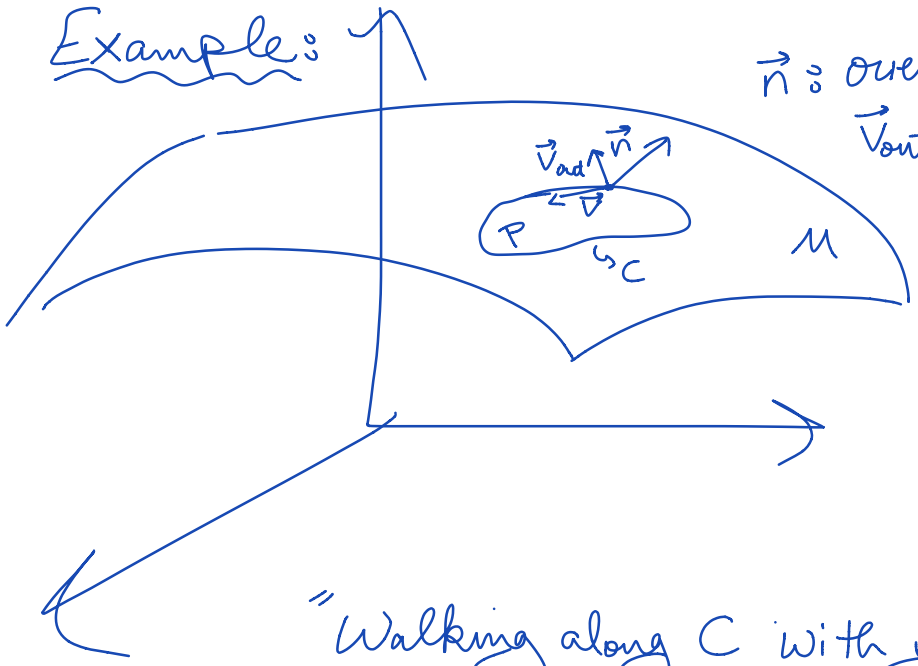
Example:



(right hand rule)

$$\Omega^{\partial}(\vec{v}) = \text{sign}\left(\det\left[\vec{v}_{out}, \vec{v}\right]\right)$$

Example:



$\vec{n}$: orients M

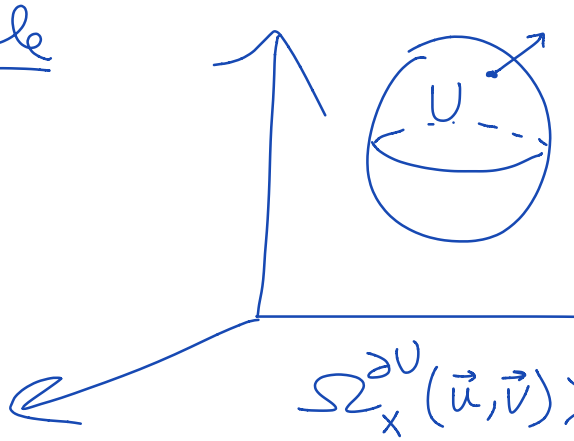
$\vec{v}_{out} \in T_x M$

$\vec{v} \in T_x \partial_M^s P$

"Walking along C with your head in the direction of $\vec{n}$, P should be on your left so that $\Omega_x^{\partial P}(\vec{v}) = +1$".

(why?)

Example



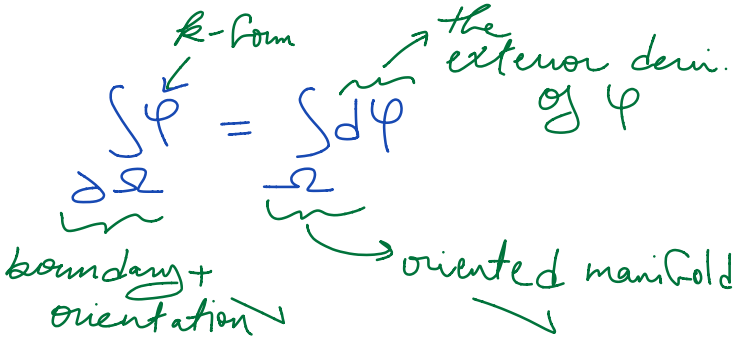
The surface of the ball in $\mathbb{R}^3$ (where $\mathbb{R}^3$ is oriented by $\Omega = \det$) is a sphere

$\Omega_x^{\partial U}(\vec{u},\vec{v}) > 0$ if $\det[\vec{u}\times\vec{v}, \vec{u}, \vec{v}] > 0$

i.e. if $\vec{u} \times \vec{v}$ is outward pointing.

6.7 The exterior derivative

Recall our goal: understanding



Defn: Let $U \subset \mathbb{R}^n$ be open. The exterior derivative

$$d: \underbrace{A^k(U)}_{k\text{-form fields}} \longrightarrow \underbrace{A^{k+1}(U)}_{k+1\text{-form fields}}$$

(*) $$(d\varphi)|_{P_x}(\vec{v}_1, \dots, \vec{v}_{k+1}) := \lim_{h\to 0} \frac{1}{h^{k+1}} \int_{\partial P_x(h\vec{v}_1, \dots, h\vec{v}_{k+1})} \varphi$$

Example when $k=0$

$$(d\varphi)_{P_x(\vec{he}_1)} = \lim_{h\to 0} \frac{1}{h} \int_{\partial P_x(h\vec{e}_1)} \varphi$$

(φ: 0-form, i.e., a function)

$\partial P_x(h\vec{e}_1)$: x & $x+h$ (boundary of a segment)

$$= \lim_{h\to 0} \frac{1}{h}[\varphi(x+h) - \varphi(x)]$$

Standard defn of derivative.

Theorem: If the k-form field φ is given by

$$\varphi = \sum_{1 < i_1 < \cdots < i_k \leq n} a_{i_1,\ldots,i_k}\, dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k}$$

where the $a_{i_1,\ldots,i_k}$'s are C^2 functions, then (*) defines a $k+1$ form.

Properties: • Let a & b be numbers & φ, ψ be k-forms

$$d(a\varphi + b\psi) = a\,d\varphi + b\,d\psi$$

- If φ is a constant form $d\varphi = 0$
- If f is a 0-form (i.e. a function)

P1

$$df = [Df] = \sum_{i=1}^{n} (D_i f)\, dx_i$$

- If f is a function

$$d(f\,dx_{i_1}\wedge\cdots\wedge dx_{i_k}) = df\wedge dx_{i_1}\wedge\cdots\wedge dx_{i_k}$$

(P_2)

Example: Compute the exterior derivative the 2-form ψ on $\mathbb{R}^4$, where

$$\psi = x_1x_2\,dx_2\wedge dx_4 - x_2^2\,dx_3\wedge dx_4$$

Sol'n: $d\psi = d\psi_1 + d\psi_2$ ← by linearity

where $\psi_1 = x_1x_2\,dx_2\wedge dx_4$, $\psi_2 = -x^2\,dx_3\wedge dx_4$

① $d\psi_1 = d(x_1x_2\,dx_2\wedge dx_4)$

$= d(x_1x_2)\wedge dx_2\wedge dx_4$ ← by P_2

$= (x_2\,dx_1 + x_1\,dx_2)\wedge dx_2\wedge dx_4$ ← by P_1 ($= \sum D_i f\,dx_i$)

$= x_2\,dx_1\wedge dx_2\wedge dx_4$

② $d\psi_2 = d(-x_2^2\, dx_3 \wedge dx_4)$

$= -d(x_2^2) \wedge dx_3 \wedge dx_4$

$= -2x_2\, dx_2 \wedge dx_3 \wedge dx_4$

$\Rightarrow d\psi = x_2\, dx_1 \wedge dx_2 \wedge dx_4 - 2x_2\, dx_2 \wedge dx_3 \wedge dx_4$

$/\!/$

Theorem: Let ψ be a C^2 k-form on $\mathbb{R}^n$:
Then $d(d\psi) = 0$

Proof: $\psi = \sum_{1 \le i_1 < \cdots < i_k \le n} f_{i_1, \ldots, i_k}\, dx_{i_1} \wedge \cdots \wedge dx_{i_k}$

Consider any of the summands

$\omega := \underbrace{f_{i_1 \cdots i_k}}_{\text{let's just call it } f}\, dx_{i_1} \wedge \cdots \wedge dx_{i_k}$

and note that $d\omega \overset{P_2}{=} df \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k}$

So $dd\omega = d(df \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k})$

$P_1 \rightarrow = d\left(\left[\sum_j D_j f\, dx_j\right] \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k}\right)$

$= d\left(\sum_j D_j f\, dx_j \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k}\right)$

linearity $= \sum_j d(D_j f) \wedge dx_j \wedge dx_{i_1} \wedge \cdots dx_{i_k})$

$P_1 \rightarrow = \left[\sum_{j=1}^{n} \sum_{k=1}^{n} D_k D_j f\, dx_k \wedge dx_j\right] \wedge dx_{i_1} \cdots \wedge dx_{i_k})$

$= 0$ (we have terms $dx_k \wedge dx_j$ & $dx_k \wedge dx_j$)

Theorem: If φ is a k-form & ψ is an l-form then

$$d(\varphi \wedge \psi) = (d\varphi) \wedge \psi + (-1)^k \varphi \wedge (d\psi)$$

Proof: Part of HW

6.8 : Grad, Curl, Div

Defn: Let $U \subset \mathbb{R}^3$ ← important be open

Let $f: U \to \mathbb{R}$ & $\vec{F}$ be a C^1 vector field.

Then ① $\operatorname{grad} f := \nabla f : \begin{bmatrix} D_1 f \\ D_2 f \\ D_3 f \end{bmatrix}$ ← gradient

② $\operatorname{curl} \vec{F} := \nabla \times \vec{F} = \begin{bmatrix} D_2 F_3 - D_3 F_2 \\ D_3 F_1 - D_1 F_3 \\ D_1 F_2 - D_2 F_1 \end{bmatrix}$

$\left(= \begin{bmatrix} D_1 \\ D_2 \\ D_3 \end{bmatrix} \times \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix} \right)$

③ $\operatorname{div} \vec{F} = \nabla \cdot \vec{F} = D_1 F_1 + D_2 F_2 + D_3 F_3$

$= \left(\vec{\nabla} \cdot \vec{F} \right)$

$\| \qquad \downarrow = (F_1\ F_2\ F_3)$

$(D_1\ D_2\ D_3)$

Example: Let $\vec{F} = (x^2, y^3, x+y+z)$

Then $\nabla \times \vec{F} = \begin{bmatrix} D_1 \\ D_2 \\ D_3 \end{bmatrix} \times \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix} = \begin{bmatrix} D_2F_3 - D_3F_2 \\ D_3F_1 - D_1F_3 \\ D_1F_2 - D_2F_1 \end{bmatrix}$

$$= \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

and $\nabla \cdot \vec{F} = 2x + 3y^2 + 1$

Theorem: (Exterior deriv. of form fields on $\mathbb{R}^3$)

1. $df = W_{\nabla f}$
2. $dW_{\vec{F}} = \mathcal{I}_{\nabla \times \vec{F}}$
3. $d\mathcal{I}_{\vec{F}} = \mathcal{M}_{\nabla \cdot \vec{F}}$

Proof of 1 $(df) = \sum_{i=1}^{3} (D_i f) \wedge dx_i$

$$\nabla f = (D_1 f, D_2 f, D_3 f)$$

$$\Rightarrow W_{\nabla f} = D_1 f \, dx_1 + D_2 f \, dx_2 + D_3 f \, dx_3 = df.$$

Proof of ②

$$d W_{\vec{F}} = d\left(F_1\, dx_1 + F_2\, dx_2 + F_3\, dx_3\right)$$

$$= dF_1 \wedge dx_1 + dF_2 \wedge dx_2 + dF_3 \wedge dx_3$$

$$= \left(\sum_{i=1}^{3} D_i F_1\, dx_i\right) \wedge dx_1 + \left(\sum_{i=1}^{3} D_i F_2\, dx_i\right) \wedge dx_2 + \left(\sum_{i=1}^{3} D_i F_3\, dx_i\right) \wedge dx_3$$

$$= \left(D_1 F_2 - D_2 F_1\right) dx_1 \wedge dx_2 + \left(D_2 F_3 - D_3 F_2\right) dx_2 \wedge dx_3 + \left(D_1 F_3 - D_3 F_1\right) dx_3 \wedge dx_1$$

$$= \Phi_{\begin{bmatrix} D_1 F_2 - D_2 F_1 \\ D_2 F_3 - D_3 F_2 \\ D_1 F_3 - D_3 F_1 \end{bmatrix}} = \Phi_{\nabla \times \vec{F}}$$

Corollary: $\nabla \times (\nabla f) = 0$

$\nabla \cdot (\nabla \times \vec{F}) = 0$

Proof of ①: $d(d\varphi) = 0$ for all forms φ

and $\begin{cases} df = W_{\nabla f} \\ dW_{\vec{F}} = \Phi_{\nabla \times \vec{F}} \end{cases}$

$\Rightarrow \underbrace{dW_{\nabla f}}_{\| } = ddf = 0$

$\Leftrightarrow \Phi_{\nabla \times (\nabla f)} = 0$ so $\nabla \times \nabla f = 0$

Proof of ② $d(d\varphi) = 0$

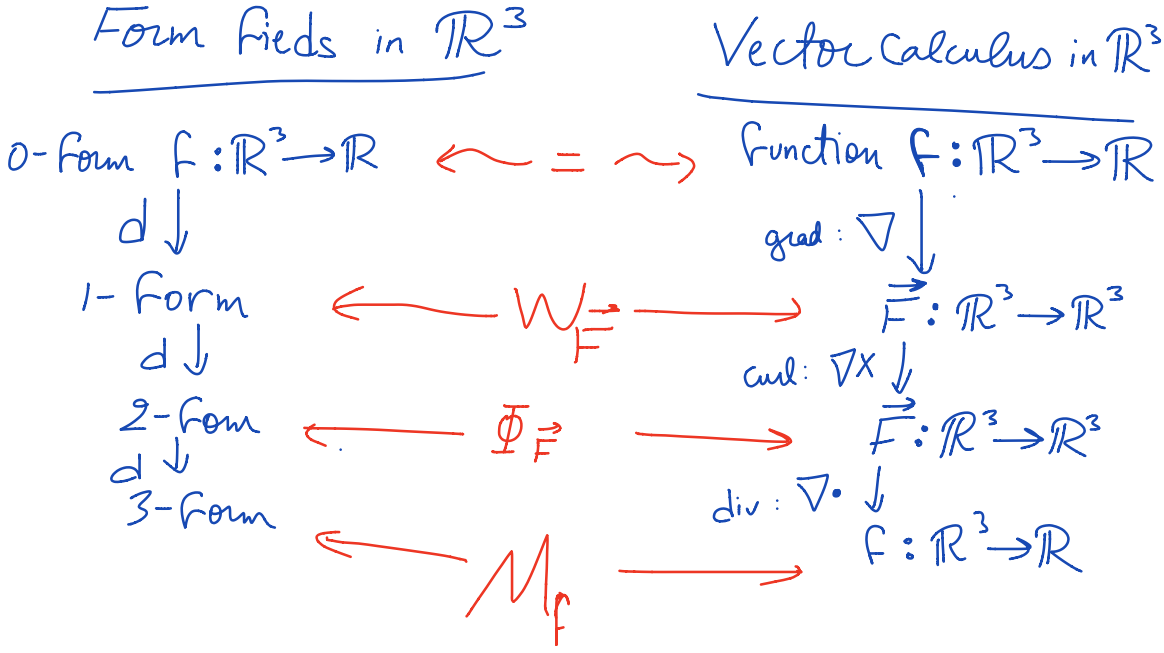
$d\Phi_{\vec{F}} = M_{\vec{\nabla} \cdot \vec{F}}$

$\Rightarrow d\Phi_{\nabla \times \vec{F}} = M_{\nabla \cdot (\nabla \times \vec{F})}$

$\Rightarrow \underbrace{ddW_{\vec{F}}}_{=0} = M_{\nabla \cdot (\nabla \times \vec{F})}$

$\Rightarrow \nabla \cdot (\nabla \times \vec{F}) = 0$

∎



6.10 The generalized Stokes Theorem

Finally!

FTOC: If f is a C^1-function

$$\int_a^b f'(t)\,dt = f(b)-f(a)$$

$$\| \qquad\qquad \|$$

$$\int_{[a,b]} df = \int_{\partial[a,b]} f$$

More Generally

Let X be a compact piece with boundary of a k-dim manifold $M \subset \mathbb{R}^n$. Give the boundary ∂X of X the boundary orientation, and let φ be a $k-1$ form defined on an open set containing X. Then

$$\int_{\partial X} \varphi = \int_X d\varphi$$

Examples: Let S be the square given by $-1 \le x \le 1$, $-1 \le y \le 1$ and let C be its boundary (oriented appropriately)

$$\int_C \overbrace{x\,dy - y\,dx}^{\varphi} = ?$$

$\partial S \nearrow C$

Stoke's

$$\int_{\partial S} x\,dy - y\,dx = \int_S d\varphi$$

$$\int_S d\varphi = \int_S dx\wedge dy - dy\wedge dx$$

$$= 2\int_S dx\wedge dy$$

$$= 2\underbrace{\int_S |dx\,dy|}_{\text{area}} = 8$$

Example: Integrate

$$\varphi = (x - y^2 + z^3)(dy\wedge dz + dx\wedge dz + dx\wedge dy)$$

over the boundary of the cube given by

$-a \le x, y, z \le a$.

Soln: denote the Cube by Ω

we want $\int_{\partial\Omega} \varphi = \int_{\Omega} d\varphi$

better than parametrizing all 6 sides

where $d\varphi = dx\wedge dy\wedge dz + 2y\,dx\wedge dy\wedge z$

$+ 3z^2\,dx\wedge dy\wedge z$

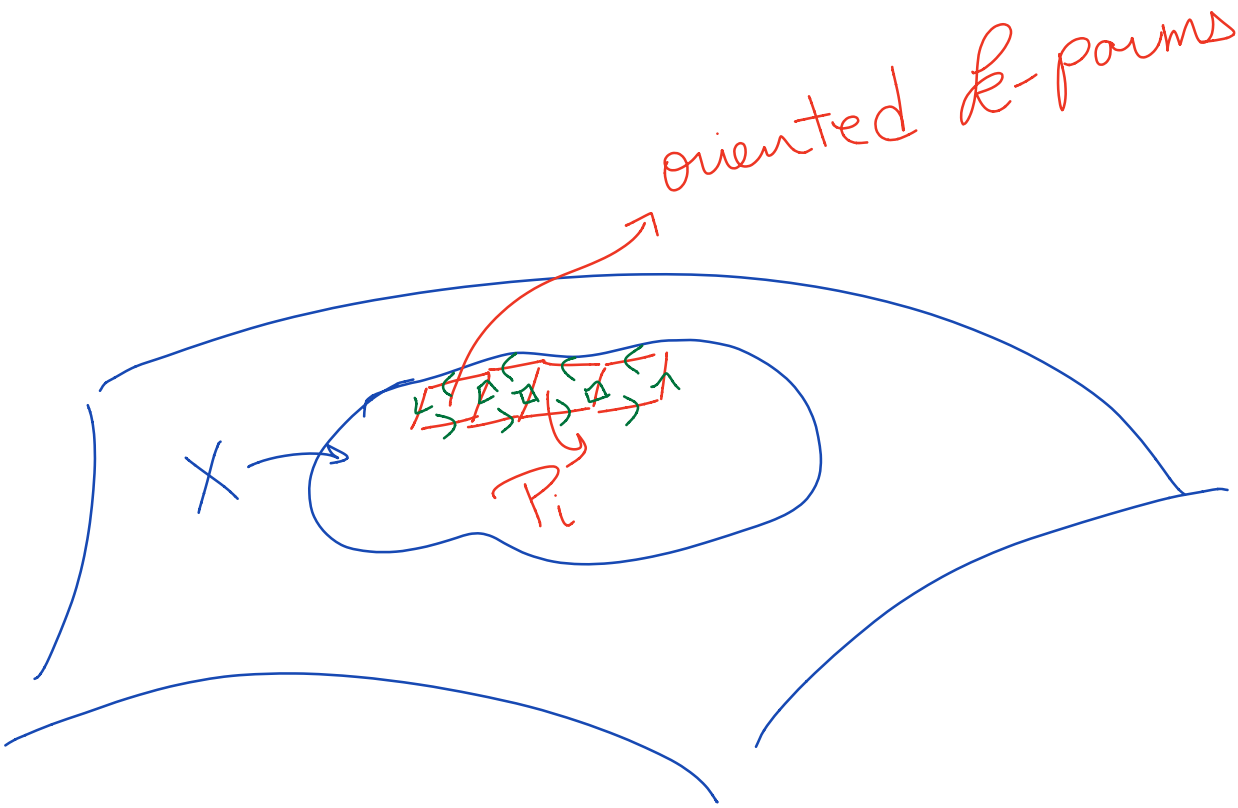
So we want $\int_\Omega (1+2y+3z^2)\, dx \wedge dy \wedge dz$

$$= \int_U (1+2y+3z^2)\, dx \wedge dy \wedge dz \, P_{(x,y,z)} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$|dx\,dy\,dz|$$

$$= \int_{-a}^{a}\int_{-a}^{a}\int_{-a}^{a} 1+2y^2+3z^2 \, dx\, dy\, dz = a^3 + a^4 + a^5$$

Informal Proof of Stokes's Theorem



$$P_i = P_{x_i}(\vec{v}_{1,i}, \ldots, \vec{v}_{k,i})$$

Recall the def'n of $d\varphi$

$$d\varphi(P_x) = \lim_{h \to 0} \frac{1}{h^{k+1}} \int_{\partial P(hv_1, \ldots hv_k)} \varphi$$

Then $\int_X d\varphi \approx \sum_i d\varphi(P_i) \approx \sum_i \int_{\partial P_i} \varphi$ ("exterior deriv. def'n")

("Riemann Sum")

$\approx \int_{\partial X} \varphi$ (cancellation of internal boundaries)

For the real proof, see A.26

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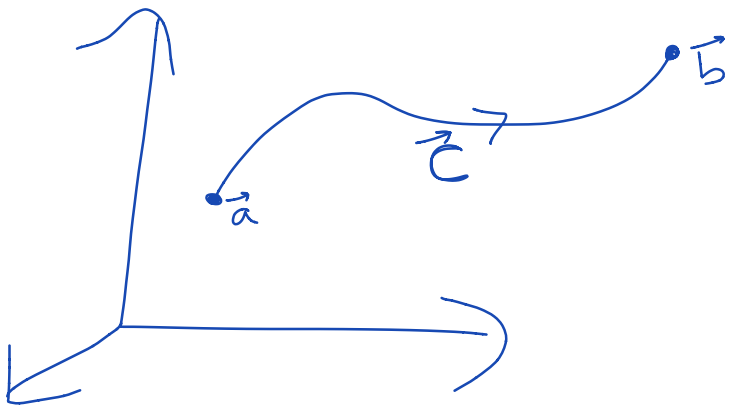
6.10 The integral Theorems of vector Calculus

Now, we're back to $\mathbb{R}^3$ (and $\mathbb{R}^2$)

1. Fund. Theorem of line integrals:



$$\int_{\vec{C}} df = f(\vec{b}) - f(\vec{a})$$

$\vec{C}$ doesn't matter

The integral only depends on the starting point and ending pt.

← 31CH

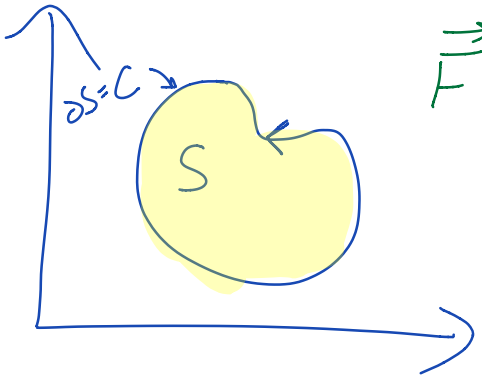
$\Updownarrow$

$$\int_{\vec{C}} (\nabla f) \cdot d\vec{s} = f(b) - f(a)$$

← 20E

Recall: $\int_{\vec{C}} \vec{F} \cdot d\vec{s} = \int_{\alpha}^{\beta} \vec{F}(\gamma(t)) \cdot \gamma'(t)\, dt$ where $\gamma(\alpha) = a, \gamma(\beta) = b$

2. Green's Theorem $(\mathbb{R}^2)$



$$\vec{F}(x,y) = f(x,y)\,\vec{i} + g(x,y)\,\vec{j}$$

$$\int_S dW_{\vec{F}} = \int_{\vec{C}} W_{\vec{F}}$$

3 ICH

$$\int_S d(f\,dx + g\,dy) = \int_{\vec{C}} f\,dx + g\,dy$$

$$\Longleftrightarrow \int_S (D_1 f\,dx + D_2 f\,dy) \wedge dx + (D_1 g\,dx + D_2 g\,dy) \wedge dy = \int_{\vec{C}} f\,dx + g\,dy$$

$$\Longleftrightarrow \int_S (D_1 g - D_2 f)\,dx \wedge dy = \int_{\vec{C}} f\,dx + g\,dy$$

$$\int_S (D_1 g - D_2 f)\,dx\,dy = \int_{\vec{C}} f\,dx + g\,dy$$

20E

Example: Evaluate, using Green's Theorem

$$\int_C y^3\,dx - x^3\,dy$$

where C is the unit circle oriented CCW.

Sol'n: $\int_C \underbrace{y^3}_{f}\,dx - \underbrace{x^3}_{g}\,dy = \iint_{\text{disk } D} \underbrace{-3x^2}_{D_1 g} \underbrace{- 3y^2}_{-D_2 f}\,dx\,dy$

$$= \int_0^{2\pi}\int_0^1 -3r^2 \cdot r\,dr\,d\theta = -6\pi/4$$

↑ check the orientation (it's fine)

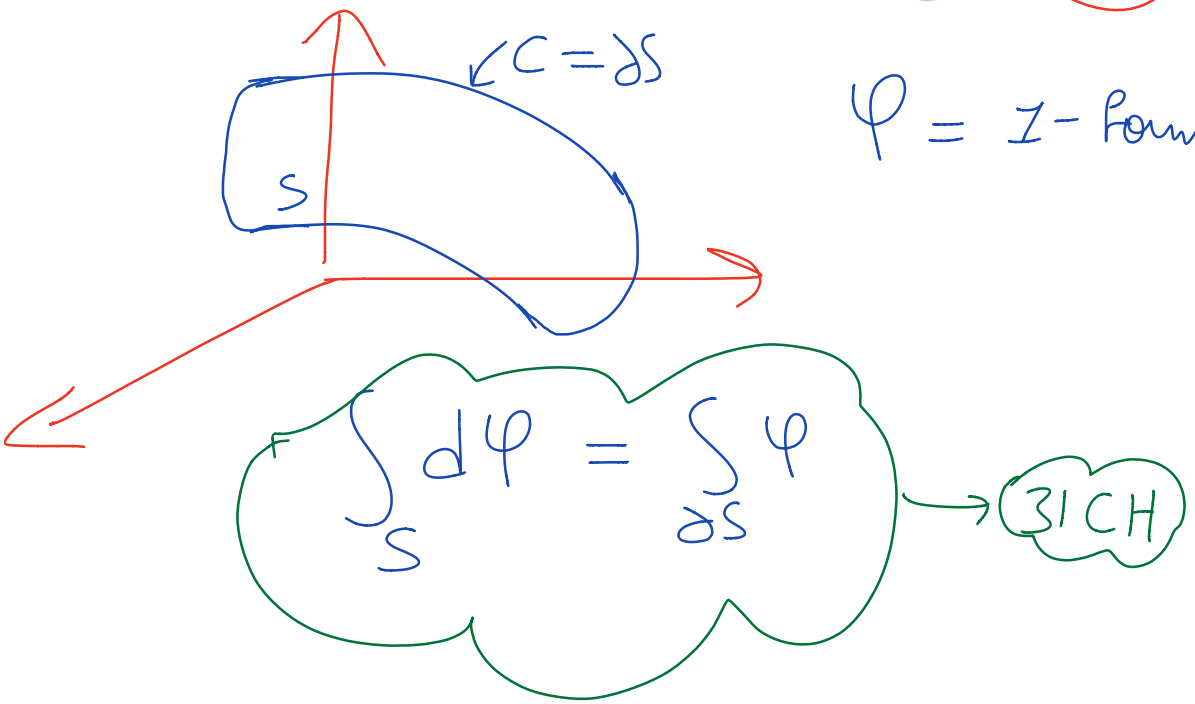
Verify, by evaluating $\int_{C^+} y^3\,dx - x^3\,dy$ directly.

Example (Area of a Region D in $\mathbb{R}^2$)

$$\iint_D dx\,dy = A(D) = \frac{1}{2}\int_{\partial D} x\,dy - y\,dx$$

Proof: Exercise

3. Stoke's Theorem (For surfaces in $\mathbb{R}^3$)



Since φ is a 1-form, we can write it as $W_{\vec{F}}$ for some vector field $\vec{F}$

So $\int_S dW_F = \int_{\partial S} W_{\vec{F}}$

$$\nabla \times F = \begin{bmatrix} D_2 F_3 - D_3 F_2 \\ D_3 F_1 - D_1 F_3 \\ D_1 F_2 - D_2 F_1 \end{bmatrix}$$

but $dW_F = \Phi_{\nabla \times F}$

So $\int_S \Phi_{\nabla \times \vec{F}} = \int_{\partial S} W_{\vec{F}}$

$\Updownarrow$

$\iint_S (\nabla \times \vec{F}) \cdot d\vec{S} = \int_{\partial S} \vec{F} \cdot d\vec{s}$ (2D)

Recall:

$$\iint_S \vec{G} \cdot d\vec{S} := \iint_T \det\left([\vec{G}, D_1\gamma, D_2\gamma]\right) du\,dv$$

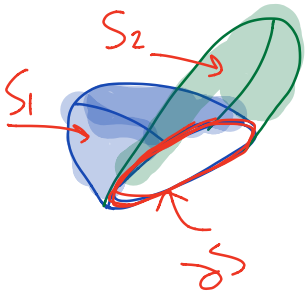
(γ: param. of S; T: domain of (u,v))

$$= \iint_T \vec{G} \cdot \left(D_1\gamma \times D_2\gamma\right)_{(u,v)} |du\,dv|$$

($D_1\gamma \times D_2\gamma$: normal to the surface)

$$= \iint_T \vec{G} \cdot \vec{N}(u,v)\,|du\,dv|$$

Example: Let S_1 & S_2 be two surfaces with the same boundary ∂S



Then

$$\iint_{S_1} \vec{F}_{\nabla \times F} = \iint_{S_2} \vec{F}_{\nabla \times F}$$

Example: Let S be the surface whose boundary is the circle $x^2+y^2=1$ where S lies above the xy-plane with a normal having +ve $\vec{k}$ component. Let

$$\vec{F} = y\vec{i} - x\vec{j} + e^{xz}\vec{k}$$

Compute $\int_S \Phi_{\nabla \times F}$ (the flux of F through S)

Sol'n: By Stokes's

$$\int_S \Phi_{\nabla \times F} = \int_{\partial S} W_{\vec{F}}$$

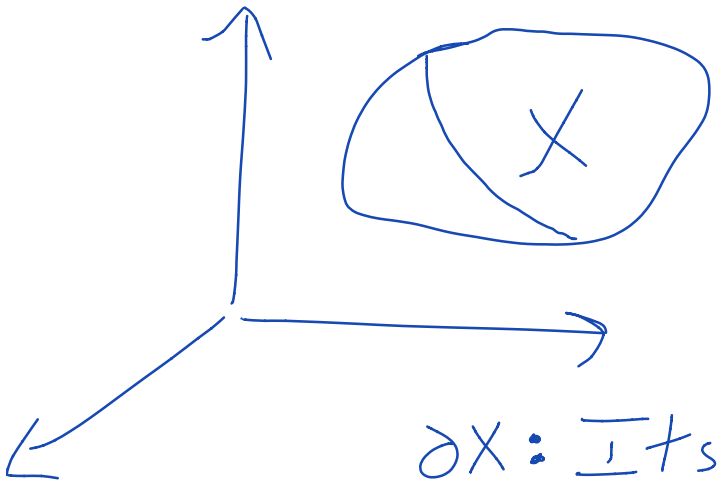
$$= \int_{[0,2\pi]} W_{\vec{F}}\left(\underbrace{P_{(\cos\theta, \sin\theta, 0)}}_{\text{ccw orientation}} \underbrace{\begin{pmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{pmatrix}}_{\gamma'(\theta)}\right) d\theta$$

$$= \int_0^{2\pi} (\sin\theta, -\cos\theta, 0) \cdot (-\sin\theta, \cos\theta, 0)\, d\theta = -2\pi$$

Example: For a surface with no boundary (e.g. the sphere)

$$\int_S \Phi_{\nabla \times F} = 0$$

4. Gauss's Divergence Thm



X: bounded domain in $\mathbb{R}^3$ (e.g. potato) with standard orientation

∂X: Its boundary (the skin of the potato) (oriented with outward normal)

φ: 2-form

$$\int_X d\varphi = \int_{\partial X} \varphi$$

but $\varphi = \Phi_{\vec{F}}$ for some $\vec{F}$

so $d\varphi = d\Phi_F = \mu_{\nabla \cdot \vec{F}}$

So $\int_X \mu_{\nabla \cdot F} = \int_{\partial X} \Phi_F$ — 3 CH

$\Uparrow \Downarrow$

$$\iiint_X \nabla \cdot F \, dx\,dy\,dz = \iint_{\partial X} \vec{F} \cdot d\vec{S}$$

2 DE

Example: $\vec{F} = 2x\vec{i} + y^2\vec{j} + z^2\vec{k}$

S is the unit sphere.

Evaluate $\iint_S \vec{F} \cdot d\vec{S}$

(the outward flux of F through S)

Solution: S is the boundary of the unit ball B so

$$\iint_S \vec{F} \cdot d\vec{S} = \int_B \nabla \cdot F$$

$$\nabla \cdot \vec{F} = 2 + 2y + 2z$$

$$\Rightarrow \int_B \nabla \cdot F = \int_B 2 + \underbrace{2y + 2z}_{\to 0 \text{ by symmetry}} \, dx\,dy\,dz$$

$$= 2\,\mathrm{Vol}(\mathrm{Ball}) = \tfrac{8}{3}\pi$$

Example Compute $\iint_S \vec{F}\cdot d\vec{S}$ where

S is the surface of the box: $0\le x\le 1$, $0\le y\le 1$, $0\le z\le 1$

and $\vec{F} = (3x + e^{\sqrt{yz}})\vec{i} + (y^2 + 5\sinh(z+x))\vec{j} + (xz + y^{-7z})\vec{k}$

Sol'n by Gauss's theorem $\iint_S \vec{F}\cdot d\vec{S} = \iiint_W \nabla\cdot\vec{F}\cdot dV$

$$= \int_0^1\int_0^1\int_0^1 \nabla\cdot\vec{F}\, dx\,dy\,dz = \int_0^1\int_0^1\int_0^1 (3+2y+x)\, dx\,dy\,dz = \int_0^1\int_0^1 (3+y+\tfrac{1}{2})\,dy\,dz$$

$$= 3.5 + 1 = 4.5$$

$$\nabla\cdot\vec{F} = \frac{\partial}{\partial x}(3x + e^{\sqrt{yz}}) + \frac{\partial}{\partial y}(y^2 + 5\sinh(z+x)) + \frac{\partial}{\partial z}(xz + y^{-7z})$$

What made this a Gauss's theorem problem?
Field was terrible but $\nabla\cdot\vec{F}$ was nice.

Applications (Physics)

Given a charge density $\rho(x,y,z)$ in a region W, the field $\vec{E}$ satisfies $\nabla\cdot\vec{E} = \rho$ (given)

$$\Rightarrow \iiint_W \nabla\cdot\vec{E}\, dV = \iint_S \vec{E}\cdot d\vec{S}$$

$\nabla\cdot\vec{E} = \rho$

$\iiint_W \nabla\cdot\vec{E}\, dV$: total charge Q inside W

$\iint_S \vec{E}\cdot d\vec{S}$: flux out of the surface

Review of main theorems

Theorem	Applies in	Applies to	States that
FTLI	2D & 3D	Curves / boundary of curve	$\int_C \nabla f \cdot d\vec{s} = f(c(b)) - f(c(a))$
Green's Theorem	2D	Regions / their boundary D / ∂D	$\iint_D (\nabla \times \vec{F}) \cdot \vec{k}\, dx\,dy = \int_{\partial D} \vec{F} \cdot d\vec{s}$ $\Leftrightarrow \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy = \int_{\partial D} P\,dx + Q\,dy$ $\Leftrightarrow \iint_D (\nabla \cdot \vec{F})\, dx\,dy = \int_{\partial D} \vec{F} \cdot \vec{n}\, ds$
Stokes's Theorem	3D	Surfaces / their boundary S / ∂S	$\iint_S (\nabla \times \vec{F}) \cdot d\vec{S} = \int_{\partial S} \vec{F} \cdot d\vec{s}$ $\Leftrightarrow \iint_S (\nabla \times \vec{F}) \cdot \vec{n}\, dS = \int \vec{F} \cdot c'(t)\, dt$
Gauss's theorem	3D	Regions / their boundary W / $\partial W = S$	$\iiint_W (\nabla \cdot \vec{F})\, dV = \iint_{\partial W} \vec{F} \cdot d\vec{S}$

6.12 Potentials

Def'n: A function $f: \mathbb{R}^n \to \mathbb{R}$ is a potential for a vector field $\vec{F}$ if $\nabla f = \vec{F}$. $(df = W_{\vec{F}})$

• A vector field $\vec{F}$ is called conservative if $\vec{F} = \nabla f$ for some function f. $(dW_{\vec{F}} = 0)$

Note: we can have $dW_{\vec{F}} = 0$,

i.e. $\nabla \times \vec{F} = 0$ without having $\vec{F} = \nabla f$ for some f.

Note: If $\vec{F}$ is conservative, i.e. $\vec{F} = \nabla f$, then $\int_{\vec{C}} \vec{F} \cdot d\vec{s}$ which is

$$\int_{\vec{C}} W_{\vec{F}} = \int_{[\gamma]} W_{\vec{F}} = f(\gamma(b)) - f(\gamma(a))$$

(Work done by conservative vector field depends on start & end points).

More generally

If ψ is a $k-1$ form we say it is a potential for $d\psi$.

If $\varphi \in A^k(U)$, we say φ admits a potential whenever $\varphi = d\psi$ for some $\psi \in A^{k-1}(U)$.

Observation $d\varphi = 0$ is a necessary condition for φ to admit a potential ($\varphi = d\psi \Rightarrow d\varphi = dd\psi = 0$)

Question: When is $d\varphi = 0$ sufficient for φ to admit a potential?

Answer:

$\forall x, y \in U, t \in [0,1]$

$tx + (1-t)y \in U$

Let $U \subset \mathbb{R}^n$ be convex and open

then $\varphi \in A^k(U)$ admits a potential

$\psi \in A^{k-1}(U)$; i.e. $\varphi = d\psi$,

if and only if $d\varphi = 0$

Poincaré's Lemma

Corollary: If $U \subset \mathbb{R}^3$ is convex and $\vec{F}$ is a C^1 vector field on U

then $\vec{F}$ is the gradient of a C^1 function

f if & only if $\nabla \times \vec{F} = 0$

Maxwell's Equations

- Physically observed laws.
- Relate $\vec{E}(x,y,z,t)$ & $\vec{H}(x,y,z,t)$ the electric & magnetic fields (time varying) to each other and to $\rho(x,y,z,t)$, the charge density and $\vec{J}(x,y,z,t)$ the current density.

The equations

$$\nabla \cdot \vec{E} = \rho/\varepsilon_0 \quad \text{(Gauss's law)} \quad (1)$$

$$\nabla \cdot \vec{H} = 0 \quad (2)$$

$$\nabla \times \vec{E} = -\frac{\partial H}{\partial t} \quad \text{(Faraday's law)} \quad (3)$$

$$\nabla \times \vec{H} = \mu_0\left(\varepsilon_0 \partial \vec{E}/\partial t + \vec{J}\right) \quad \text{(Ampere's law)} \quad (4)$$

(μ_0: permeability; ε_0: permittivity)

- For example (1) tells us that $\iiint_W \rho \, dS$ = total charge = $\iint_{\partial W} \vec{E} \cdot d\vec{S}$
- (2) has no RHS because there are no free magnetic charges
- (3) & (4) tell us that a time varying $\vec{E}$ lectric field induces a magnetic field & vice versa.

Let's use the language of vector calculus to get a sense of what Maxwell's equations can imply (see 6.11 for connections to forms and for much more detail)

Let's look at the equations in vacuum i.e. $\vec{J}=0$, $\rho=0$

$$\nabla\times\vec{E}=-\frac{\partial H}{\partial t} \quad \& \quad \nabla\times\vec{H}=\mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t}$$

So

$$\nabla\times(\nabla\times\vec{E})=-\frac{\partial(\nabla\times H)}{\partial t}=-\partial/\partial t\,(\mu_0\varepsilon_0\,\partial\vec{E}/\partial t)=-\mu_0\varepsilon_0\,\partial^2\vec{E}/\partial t^2$$

$\nabla^2(E_x)\vec{i}+\nabla^2(E_y)\vec{j}+\nabla^2(E_z)\vec{k}$ (vector laplacian)

but $\nabla\times(\nabla\times\vec{E})=\nabla(\nabla\cdot\vec{E})-\nabla^2\vec{E}$ (see section 4.4)

So $\nabla(\nabla\cdot\vec{E})-(\partial^2\vec{E}/\partial x^2+\partial^2\vec{E}/\partial y^2+\partial^2\vec{E}/\partial z^2)=-\mu_0\varepsilon_0\,\partial^2\vec{E}/\partial t^2$

$\Rightarrow \mu_0\varepsilon_0\,\partial^2\vec{E}/\partial t^2=\nabla^2\vec{E}$

$\frac{1}{c^2}$

Similarly $\frac{1}{c^2}\partial^2H/\partial t^2=\nabla^2\vec{H}$

So EM waves propagate at the speed of light (Maxwell suspected) and light is an EM wave!!